Elements of price index number theory

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1 Introduction

In these brief notes, we introduce some basic elements of the theory of price and quantity indexes, which constitutes one of the main building blocks of economic statistics. The basic aim of the theory is to provide tools for comparing economic aggregates over time and over space on a "common monetary unit". For example, consider comparing the national GDP over two times $t_0 < t_1$, with the aim to check whether the economy is expanding or contracting. The direct comparison of the GDP values is not meaningful since, going from t_0 to t_1, both the quantities *and the prices* of the produced goods and services change and one cannot understand whether the change in the value is *real* or *nominal*. To make the comparison meaningful, we must remove the price component from the change of the GDP value, i.e. we must express the GDP at time t_1 *at the prices of time* t_0 (or viceversa). To do this, we must define a so-called *price index*, to be used to make two different aggregates (the GDPs at different times) expressed in an *equivalent currency*. As we shall see, this is not that straightforward and requires a careful definition of the properties of the formulas used to build the index. Similar problems must be addressed when one wants to compare the value of the same basket of goods in two different countries, with different currencies. It is not enough to use currency exchange rates, to convert the aggregates to the same monetary unit, since exchange rates do not express the relation between the internal *purchasing powers* of the currencies. So it is necessary to build a specific index, capable to convert the currencies to a common unit, in such a way to clean their purchasing power, from nominal differences.

It is thus clear that, without common monetary units, any economic analysis involving values, costs, quantities... and their comparison over time and space has no meaning and this is why price index theory is a key tool for economic statistics.

There are different ways to introduce the theory; we will follow the so-called *axiomatic* approach which has the advantage to clarify the logic thread of index construction and to formalize it neatly. The mathematical effort required by this approach pays back, in terms of greater synthesis, conceptual unity and technical soundness.

Remark 1. Sections marked with "*" are to be understood as technical material, necessary to understand the main content, but not required for the exam.

Remark 2. All numerical examples are taken from Martini M. (2003), *Numeri indice per il confronto nel tempo e nello spazio*, Cusl.

Remark 3. These notes are just a support to be completed with the comments provided during the frontal lessons; they are not to be meant as a complete textbook on this part of the course.

2 Technical preliminaries*

In this section, we introduce some technical definitions and results which prove useful in the development of price index theory.

2.1 Partially ordered sets*

Consider a basket of n goods with their associated vectors of prices $\boldsymbol{p}_a = (p_{a1}, \ldots, p_{an})$ and $\boldsymbol{p}_b = (p_{b1}, \ldots, p_{bn})$, in two different "situations" a and b (e.g., at two different times, or in two different towns and the like). If prices in a are all not higher than prices in b, i.e. if $p_{ai} \leq p_{bi} \quad \forall\, i = 1, \ldots, n$, then we can globally say that the two vectors are *comparable* and that price vector $\boldsymbol{p}_a$ is not higher than price vector $\boldsymbol{p}_b$ (written $\boldsymbol{p}_a \trianglelefteq \boldsymbol{p}_b$). However, if some prices in a are lower than in b and some prices in b are lower than in a, i.e. if there are *conflicting* prices, then the two price vectors are *incomparable* (written $\boldsymbol{p}_a || \boldsymbol{p}_b$). So, in general, price vectors can be ordered only *partially* (i.e. in some cases thay can be ordered, in others they cannot). Now, consider the set Π of all price vectors on the same n goods and endow it with the order relation $\trianglelefteq$ defined above. The pair $(\Pi, \trianglelefteq)$ is called a *partially ordered set* or a *poset*, for short. When all the elements of a poset can be ordered (i.e. when the partial order is in fact not partial…), it is called a *total* (or *complete*, or *linear*) order. For example, the set of real numbers is a totally ordered set. Suppose we have two posets $(\Lambda_1, \trianglelefteq_1)$ and $(\Lambda_2, \trianglelefteq_2)$ and a function $f(\cdot)$, mapping elements of Λ_1 to elements of Λ_2. If $x_1 \trianglelefteq_1 x_2$ implies $f(x_1) \trianglelefteq_2 f(x_2)$, then function $f(\cdot)$ is said to be *order preserving.* As we will see, price indexes are order preserving maps from the poset of price vectors, to the totally ordered set of positive real numbers.

2.2 Aggregation systems*

In this paragraph, we discuss the concept of "associativity", which plays a role in price index theory. Suppose you compare the prices of a basket of goods (considered as a whole) over time and also the prices of its "components". It is natural to ask for a sort of "consistency" between the results you get in the first case and those you get in the second case. So you want, somehow, a way of aggregating price variations which is "independent" of the "observation scale". This leads to the study of aggregative systems and of their mathematical representations.

Let $\boldsymbol{x} = (x_1, \ldots, x_k)$ be a vector of k real numbers in $[0, 1]$, $\boldsymbol{w} = (w_1, \ldots, w_k)$ a vector of k non-negative weights summing to 1 and $g(\cdot)$ a continuous and strictly monotone real function, from $[0,1]$ to $\mathbb{R}$, called the *generating* function. The *weighted quasi-arithmetic mean* $M_{g,k}(\cdot)$ is defined as:

$$M_{g,k}(\boldsymbol{x};\boldsymbol{w}) = g^{-1}\left(\sum_{i=1}^{k} w_i g(x_i)\right) \tag{1}$$

where $g^{-1}(\cdot)$ is the inverse of function $g(\cdot)$ which exists and is well-defined from $\mathbb{R}$ to $[0,1]$, since $g(\cdot)$ is strictly monotone. Many well-known means belong to the class of quasi-arithmetic means, namely the *weighted power means* $M_{[r],k}(\cdot)$:

$$M_{[r],k}(\boldsymbol{x};\boldsymbol{w}) = \left(\sum_{i=1}^{k} w_i x_i^r\right)^{1/r} \quad (g(x) = x^r,\ r \neq 0); \tag{2}$$

$$M_{[0],k}(\boldsymbol{x};\boldsymbol{w}) = \prod_{i=1}^{k} x_i^{w_i} \quad (g(x) = \log(x)). \tag{3}$$

When $\boldsymbol{w} = (1/k,\ldots,1/k)$, the above formulas reduce to "classical" means (called *quasi-arithmetic means*):

$$M_{g,k}(\boldsymbol{x}) = g^{-1}\left(\frac{1}{k}\sum_{i=1}^{k} g(x_i)\right); \tag{4}$$

$$M_{[r],k}(\boldsymbol{x}) = \left(\frac{1}{k}\sum_{i=1}^{k} x_i^r\right)^{1/r}; \tag{5}$$

$$M_{[0],k}(\boldsymbol{x}) = \left(\prod_{i=1}^{k} x_i\right)^{\frac{1}{k}}. \tag{6}$$

What makes quasi-arithmetic means of particular relevance for our purposes is that they contain the class of *continuous*, *strictly monotone* and *decomposable* functionals. To be formal, let us call *aggregation system* a collection $\mathbb{F} = \{F_1(\cdot), F_2(\cdot), F_3(\cdot)\ldots\}$ of functionals with $1, 2, 3\ldots$ arguments respectively, with $F_1(x) = x$ by convention. Then continuity, strict monotonicity and decomposability for $\mathbb{F}$ are defined as follows:

1. **Continuity**. An aggregation system $\mathbb{F}$ is continuous if each of its elements $F_k(\cdot) \in \mathbb{F}$ is continuous in each of its k arguments ($k = 1, 2, \ldots$).

2. **Strict monotonicity**. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two k-dimensional real vectors; a functional $F_k(\cdot)$ is strictly monotone if $\boldsymbol{x} \trianglelefteq \boldsymbol{y}$ in $\mathbb{R}^k$ implies $F_k(\boldsymbol{x}) < F_k(\boldsymbol{y})$. An aggregation system $\mathbb{F}$ is strictly monotone if each of its elements is strictly monotone.

3. **Decomposability**. An aggregation system $\mathbb{F}$ is decomposable if and only if for all $m, n = 1, 2, \ldots$ and for all $\boldsymbol{x} \in [0,1]^m$ and $\boldsymbol{y} \in [0,1]^n$:

$$F_{m+n}(\boldsymbol{x}, \boldsymbol{y}) = F_{m+n}(\underbrace{F_m(\boldsymbol{x}_m), \ldots, F_m(\boldsymbol{x}_m)}_{m\ times}, \boldsymbol{y}). \tag{7}$$

The last formula means that the value of functional $F_{m+n}(\cdot)$ can be obtained substituting to the first m arguments their aggregated value $F_m(\cdot)$, replicated m times. The statement becomes clearer when specialized to arithmetic means. In this case, it simply states that one can compute the average of $m+n$ numbers substituting to each of the first m the average of the first m numbers themselves.

According to the Nagumo-Kolmogorov theorem, an aggregation system $\mathbb{F} = \{F_k(\cdot)\}$ $(k = 1, 2, 3, \ldots)$ is continuous, strictly monotone and decomposable if and only if there exists a monotone bijective function $g(\cdot) : [0,1] \to [0,1]$ such that for $k > 1$, $F_k(\cdot)$ is a quasi-arithmetic mean $M_{g,k}(\cdot)$. A functional F is *homogeneous* if, for every real number $c \in [0,1]$, it is $F(c \cdot \boldsymbol{x}) = c \cdot F(\boldsymbol{x})$; it can be proved that the only homogeneous quasi-arithmetic means are the power means $M_{[r],k}(\cdot)$. Finally, notice that quasi-arithmetic means are *symmetric*, i.e. they are invariant under permutations of their arguments. As a consequence, they satisfy the property of *strong decomposability*, i.e they are invariant under the aggregation of any subset of (and not just of consecutive) arguments.

Now, let $\mathbb{F}$ be an aggregation system. Assume its elements are symmetric (as defined above) and suppose that, if vector $\boldsymbol{x} = (x_1, \ldots, x_m)$ is partitioned into k subvectors $\boldsymbol{x}_{(1)}, \ldots, \boldsymbol{x}_{(k)}$, of length $n_1, \ldots, n_k$ respectively, it holds:

$$F_m(\boldsymbol{x}) = F_k\left(F_{n_1}(\boldsymbol{x}_{(1)}), \ldots, F_{n_k}(\boldsymbol{x}_{(k)})\right) \tag{8}$$

(i.e. suppose that aggregation can be performed "aggregating partial aggregations"). An aggregation system satisfying (9) will be called *consistent-in-aggregation*. An important special case of the above formula is the following

$$F_m(x_1, \ldots, x_m) = F_2\left(F_{m-1}(x_1, \ldots, x_{m-1}), x_m\right), \tag{9}$$

which means that vector $\boldsymbol{x}$ can be aggregated in "two steps", the first of which aggregates $m-1$ components. Using this formula repeatedly, one can reduce $F_m(\cdot)$ to a nested sequence of applications of $F_2(\cdot)$; for example:

$$F_4(x_1, x_2, x_3, x_4) = F_2\left(F_3(x_1, x_2, x_3), x_4\right) = F_2\left(F_2(F_2(x_1, x_2), x_3), x_4\right). \tag{10}$$

One thus sees that $F_2(\cdot,\cdot)$ determines the entire aggregation system $\mathbb{F}$. Thanks to symmetry, $F_2(x_1, x_2) = F(x_2, x_1)$ and also $F_2\left(F_2(x_1, x_2), x_3\right) = F_2\left(x_1, F_2(x_2, x_3)\right)$, i.e. $F_2(\cdot,\cdot)$ is *commutative* and *associative*. Thus $F_2(\cdot,\cdot)$ is a[1] *commutative semigroup operation* and

[1]A *semigroup* is a set on which it is defined a binary - i.e. acting on two elements - associative operation.

$\mathbb{F}$ is a *commutative semigroup*, *generated* by $F_2(\cdot,\cdot)$. Denoting $F_2(\cdot,\cdot)$ as $\circ_F$, we can write formula (11) in the following clearer way

$$F_4(x_1,x_2,x_3,x_4) = (x_1 \circ_F x_2) \circ_F x_3) \circ_F x_4 = x_1 \circ_F x_2 \circ_F x_3 \circ_F x_4, \tag{11}$$

where the second equality comes from associativity of $\circ_F$.

For our purposes, what is interesting is that weighted quasi-arithmetic means, and power means in particular, are consistent-in-aggregation and are generated by a suitable choice of $F_2(\cdot,\cdot)$. To show this, some notation must be introduced first. Let x_1 and x_2 be the numbers we want to aggregate, using a weighted quasi-arithmetic mean with weights w_1 and w_2. Put $\boldsymbol{x_1} = (x_1, w_1)$ and $\boldsymbol{x_2} = (x_2, w_2)$; then the following binary operation $\circ_g$ generates $M_{g,k}(\cdot)$:

$$\boldsymbol{x}_1 \circ_g \boldsymbol{x}_2 = \left(g^{-1}\left(\frac{w_1 g(x_1) + w_2 g(x_2)}{w_1 + w_2}\right); w_1 + w_2\right) \tag{12}$$

Specializing this formula to $g(\cdot) = id(\cdot)$ (identity function) we get the generating operator for the weighted arithmetic mean:

$$\boldsymbol{x}_1 \circ_{id} \boldsymbol{x}_2 = \left(\frac{w_1 x_1 + w_2 x_2}{w_1 + w_2}; w_1 + w_2\right). \tag{13}$$

Applying recursively this formula to a set of numbers $x_1, \ldots, x_m$, starting with $w_1 = w_2 = 1$, gives the simple arithmetic mean.

Remark. Notice that to represent weighted and unweighted quasi-arithmetic means in semigroup terms, it has been necessary to jointly state the update formula for both the values and the weights, so as that each step of the recursion carries over all of the information needed for the next nested application of the semigroup operation.

3 Temporal price indexes

The main goal of price indexes is to compare price over time, so as to get a measure of price variation (*inflation/deflation*) and to turn *nominal* measures of economic aggregates into *real* ones. Temporal comparisons can be *bilateral*, i.e. they can involve the prices of the same goods at two different times, or *multilateral*, tracing the dynamics of prices over time, in a time series. We first address bilateral price indexes and then turn to the multitemporal case.

3.1 Bilateral price indexes

Given two times $t_0 < t_1$, and a fixed basket of goods, we want to compare their prices to get a synthetic measure of price variation. Such a *bilateral* comparison can be performed considering just the prices of the goods or their quantities (e.g. consumed or produced) too. In the first case, price indexes are called *simple*, in the second *composed*. The theory of simple price index is much simpler than that of the composed ones, even if in real applications, composed indexes are of greater interest (e.g. official inflation measures do consider quantities).

3.1.1 Simple price indexes

Let $\boldsymbol{p}_0 = (p_{01}, \ldots, p_{0n})$ and $\boldsymbol{p}_1 = (p_{11}, \ldots, p_{1n})$ be the price vectors of a basket of n goods, at time t_0 and t_1 respectively. A simple price index is just a positive function $P(\cdot, \cdot)$, "collapsing" pairs of price vectors into a single positive number:

$$\begin{aligned} P \;&:\; \mathbb{R}^{2n} \mapsto \mathbb{R}^{+} \\ &:\; (\boldsymbol{p}_1, \boldsymbol{p}_0) \to P(\boldsymbol{p}_1, \boldsymbol{p}_0). \end{aligned}$$

In principle, many different functions could work to measure price variations. For example, one could compute the averages $\overline{\boldsymbol{p}}_0$ and $\overline{\boldsymbol{p}}_1$ of prices at t_0 and t_1 and then consider the ratio $\overline{\boldsymbol{p}}_1/\overline{\boldsymbol{p}}_0$; alternatively, we could compute the ratio $r_i = p_{1i}/p_{0i}$ of the prices of each single good, at t_1 and t_0, and then consider the average ratio $(r_1 + \ldots + r_n)/n$. In order to restrict the family of eligible functions, and to disregard unuseful ones, we follow an axiomatic approach and define a list of "reasonable" properties that a function has to fulfill, in order to be accepted as a simple price index. The axiom list is the following:

- **Proportionality** (PR). $P(\lambda \boldsymbol{p}_0, \boldsymbol{p}_0) = \lambda \quad (\forall\, \lambda > 0)$. Proportionality requires that if t_1-prices are proportional to t_0-prices, then the price index must be equal to the constant of proportionality.

- **Commensurability** (C). $P(\boldsymbol{p}_1 \cdot U, \boldsymbol{p}_0 \cdot U) = P(\boldsymbol{p}_1, \boldsymbol{p}_0)$, for any diagonal matrix U with diagonal elements different from 0. Commensurability requires that the price index is independent of the measure unit adopted (e.g. euros, dollars. . .) in each elementary price comparison.

- **Homogeneity** (H). $P(\lambda_1 \boldsymbol{p}_1, \lambda_0 \boldsymbol{p}_0) = \lambda_1/\lambda_0 P(\boldsymbol{p}_1, \boldsymbol{p}_0)$, for $\lambda_1, \lambda_0 > 0$. Homogeneity requires the index to scale linearly in λ_1, when t_1 prices are multiplied by λ_1, and linearly in $1/\lambda_0$, when t_0 prices are multiplied by λ_0. The idea is that if prices in t_1 all grow proportionally, the price index must increase with the same proportionality constant (and, analogously, decreses if prices in t_0 decrease proportionally).

- **Monotonicity** (M). If $\boldsymbol{p}_1 \trianglelefteq \boldsymbol{p}_1^*$, then $P(\boldsymbol{p}_1, \boldsymbol{p}_0) \leq P(\boldsymbol{p}_1^*, \boldsymbol{p}_0)$. Monotonicity requires price indexes (for fixed $\boldsymbol{p}_0$) to be *order preserving* maps, from the partial order of price vectors to the (complete) order of non-negative real numbers. This property is natural, since we want the price index to grow when prices increase.

- **Associativity** (A). Price indexes must be consistent-in-aggegation functions of elementary prices (where aggregation must involve the same subgroups of indexes, both in vectors $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$). This property requires to be the same building an index on a set of goods or by first splitting them into "sub-goods" (or, symmetrically, by aggregating them in "super-goods"). Notice that to be coherent with the definition of consistency-in-aggregation, we should define a price index as a family of functions. For simplicity's sake, we drop such a formal issue, being clear what is meant by "associativity" in this case.

- **Basis reversibility** (B). $P(\boldsymbol{p}_1, \boldsymbol{p}_0) = 1/P(\boldsymbol{p}_0, \boldsymbol{p}_1)$. Basis reversibility requires that exchanging t_0 and t_1 is the same as taking the reciprocal of the price index (mimicing what happens for elementary price ratios). By defining the *basis antithesis* $B(P) = 1/P(\boldsymbol{p}_0, \boldsymbol{p}_1)$, basis reversibility can be stated as the *invariance* property $B(P) = P$.

The main result, in simple price theory, is that the above requirements uniquely identify the price index formula. In fact:

1. Putting $U_{ii} = p_{0i}^{-1}$, by commensurability we have

$$P(\boldsymbol{p}_1, \boldsymbol{p}_0) = P(\boldsymbol{p}_1/\boldsymbol{p}_0, \mathbf{1}), \tag{14}$$

   where $\boldsymbol{p}_1/\boldsymbol{p}_0$ is the vector of price ratios p_{1i}/p_{0i} and $\mathbf{1}$ is the vector all of whose compontents are equal to 1. In other words, commensurability implies the price index to be a function[2] of price ratios (and we can drop the dependance by the constant vector $\mathbf{1}$).

2. By proportionality we then have $P(\lambda, \ldots, \lambda) = \lambda$, i.e. P is a *consistent* function.

3. Homogeneity, monotonicity and associativity directly transfer to price ratios so that, in the end, a price index is required to be a *consistent, monotonic, associative and homogeneous* function of price ratios.

4. The Kolmogorov-Nagumo Theorem, with de Finetti's corollary, states that the only functions with these properties are the power means, so that

$$P(\boldsymbol{p}_1, \boldsymbol{p}_0) = \left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{p_{1i}}{p_{0i}}\right)^s\right]^{1/s} \tag{15}$$

   where

$$\lim_{s\to 0}\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{p_{1i}}{p_{0i}}\right)^s\right]^{1/s} = \left[\prod_{i=1}^{n}\frac{p_{1i}}{p_{0i}}\right]^{1/n} \quad \text{(geometric mean).} \tag{16}$$

5. Among the above formulas, the only one fulfilling basis reversibility (B) is the last one. To see this, consider that trivially $s = 0 \Rightarrow (B)$; on the other hand, let $n = 2$ and let $r_1 = p_{11}/p_{01}$ and $r_2 = p_{12}/p_{02}$. In this case, the basis reversibility condition reads:

[2] To be formal, we should denote such a function differently, for example by $\hat{P}$, but to avoid complicated notations, we use the same symbol P.

$$\frac{1}{2}(r_1^s + r_2^s) = 2\left[\left(\frac{1}{r_1}\right)^s + \left(\frac{1}{r_2}\right)^s\right]^{-1}. \tag{17}$$

Now let $r_1 = 1$ and let us consider the limits of the two sides as $r_2 \to +\infty$; for $s > 0$ we have

$$\lim_{r_2 \to +\infty} 1 + r_2^s = +\infty; \lim_{r_2 \to +\infty} 4\left[1 + \left(\frac{1}{r_2}\right)^s\right]^{-1} = 4 \tag{18}$$

while for $s < 0$ it is

$$\lim_{r_2 \to +\infty} 1 + r_2^s = 1; \lim_{r_2 \to +\infty} 4\left[1 + \left(\frac{1}{r_2}\right)^s\right]^{-1} = 0. \tag{19}$$

In any case, the limits of the two sides are different, so for any value of $s \neq 0$ there is a value r_2^* such that the two expressions are different, for any $r_2 > r_2^*$. We have thus proved that $(B) \Rightarrow s = 0$.

In summary, the only axiomatically acceptable simple price index is the geometric mean of price ratios, known as the **Jevons index**.

Remark 1. Notice that the Jevons index can be defined as the geometric mean of the price ratios or as the ratio of the geometric means of the prices at t_0 and t_1, respectively.

Remark 2. In practical applications, for example when measuring inflation, it is clear tha quantities matter (e.g., the impact of the increase of the price of bread is much more relevant to people, than the increase in the price of a much less consumed good). However, in some cases we do not have information on quantities and so we must employ the Jevons index. As it will be discussed later, the measurement of inflation is based on a "pyramidal aggregation"process, starting from a high number of products, successively aggregated in larger classes. Since quantities are not available at the most basic level, the Jevons index is used in the production of the "first layer" of price indexes.

3.2 Composite price indexes

Differently from the simple ones, composite price indexes build price comparisons by taking into account both the prices and the quantities of the basket of goods at hand.

Let $\boldsymbol{p}_0 = (p_{01}, \ldots, p_{0n})$ and $\boldsymbol{p}_1 = (p_{11}, \ldots, p_{1n})$ be the price vectors of a basket of n goods, at time t_0 and t_1 respectively, and let $\boldsymbol{q}_0 = (q_{01}, \ldots, q_{0n})$ and $\boldsymbol{q}_1 = (q_{11}, \ldots, q_{1n})$ be the corresponding vectors of quantities (e.g. consumed or produced). The *value index* V is simpy defined as:

$$V(\boldsymbol{p}_1, \boldsymbol{p}_0, \boldsymbol{q}_1, \boldsymbol{q}_0) = \frac{\sum_{1=1}^n p_{1i}q_{1i}}{\sum_{1=1}^n p_{0i}q_{0i}} \tag{20}$$

i.e. as the ratio between the value (sum of price × quantities) at time t_1 and the value at time t_0. The variation of the value over the two times may be due to a variation of (some) prices and/or of (some) quantities and we want to measure such two contributions separately. Formally, we want to decompose the value index into the product of two indexes $P(\boldsymbol{p}_1, \boldsymbol{p}_0, \boldsymbol{q}_1, \boldsymbol{q}_0)$ and $Q(\boldsymbol{p}_1, \boldsymbol{p}_0, \boldsymbol{q}_1, \boldsymbol{q}_0)$, where the first can be interpreted as a price index and the second as a quantity index:

$$V = P(\boldsymbol{p}_1, \boldsymbol{p}_0, \boldsymbol{q}_1, \boldsymbol{q}_0) \cdot Q(\boldsymbol{p}_1, \boldsymbol{p}_0, \boldsymbol{q}_1, \boldsymbol{q}_0). \tag{21}$$

Roughly speaking, the idea is that the basket of goods is seen as a "global" entity, whose "global" value change is given by the product of a "global" price variation and a "global" quantity variation, as in the case of a single good.

What makes $P(\boldsymbol{p}_1, \boldsymbol{p}_0, \boldsymbol{q}_1, \boldsymbol{q}_0)$ and $Q(\boldsymbol{p}_1, \boldsymbol{p}_0, \boldsymbol{q}_1, \boldsymbol{q}_0)$ a price index and a quantity index, respectively? In other words, what distinguishes the two functions and what features must they have in order to be interpreted as a measure of price variation and of quantity variation, respectively? This leads to defining a set of axioms that functions P and Q must fulfill, to be considered price and quantity indexes. Since our main interest is on prices, we state price index axioms, which are partly analogous to those stated in the simple case (the axioms for quantity indexes are obtained just exchanging price and quantity vectors in the price index axioms). We split the list into "mandatory axioms" (i.e. axioms that cannot be violated, otherwise the functions cannot be interpreted consistently as price indexes) and "desirable axioms" (i.e. properties that are not essential, but nevertheless relevant if enjoied).

Mandatory properties

1. **Proportionality** (P). $P(\lambda \boldsymbol{p}_0, \boldsymbol{p}_0, \boldsymbol{q}_1, \boldsymbol{q}_0) = \lambda$ ($\lambda > 0$). This property requires that, independently of the quantities, if the price vectors are proportional, the index must be equal to the constant of proportionality. This property is essential, since it defines the "nature" of the index, in this case that the index actually measures price variation.

2. **Commensurability** (C). $P(\boldsymbol{p}_1 \cdot U, \boldsymbol{p}_0 \cdot U, \boldsymbol{q}_1 \cdot U^{-1}, \boldsymbol{q}_0 \cdot U^{-1}) = P(\boldsymbol{p}_1, \boldsymbol{p}_0, \boldsymbol{q}_1\, \boldsymbol{q}_0)$, for any diagonal matrix U with diagonal elements different from 0.

3. **Homogeneity** (H). $P(\lambda_1 \boldsymbol{p}_1, \lambda_0 \boldsymbol{p}_0, \boldsymbol{q}_1, \boldsymbol{q}_0) = \lambda_1/\lambda_0 P(\boldsymbol{p}_1, \boldsymbol{p}_0, \boldsymbol{q}_1, \boldsymbol{q}_0)$, for $\lambda_1, \lambda_0 > 0$. Homogeneity requires the index to scale linearly in λ_1, when t_1 prices are multiplied by λ_1, and linearly in $1/\lambda_0$, when t_0 prices are multiplied by λ_0.

4. **Monotonicity** (M). If $\boldsymbol{p}_1 \trianglelefteq \boldsymbol{p}_1^*$, then $P(\boldsymbol{p}_1, \boldsymbol{p}_0, \boldsymbol{q}_1, \boldsymbol{q}_0) \leq P(\boldsymbol{p}_1^*, \boldsymbol{p}_0, \boldsymbol{q}_1, \boldsymbol{q}_0)$. Monotonicity requires price indexes (for fixed $\boldsymbol{p}_0, \boldsymbol{q}_0, \boldsymbol{q}_1$) to be *order preserving* maps, from the partial order of price vectors to the (complete) order of non-negative real numbers. This property is natural, since we want the price index to grow when prices increases.

Desirable properties

1. **Basis reversibility** (B). $P(\boldsymbol{p}_1, \boldsymbol{p}_0, \boldsymbol{q}_1, \boldsymbol{q}_0) = P(\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{q}_0, \boldsymbol{q}_1)^{-1}$. This requirement is completely analogous to that stated for simple price indexes. Calling $B(P) = P(\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{q}_0, \boldsymbol{q}_1)^{-1}$ the *basis antithesis* of P, this axiom simply requires the index P to coincide with its basis antithesis, i.e. to be *invariant* under the basis reversion operation B.

2. **Factor reversibility** (F). $V(\boldsymbol{p}_1, \boldsymbol{p}_0, \boldsymbol{q}_1, \boldsymbol{q}_0)/P(\boldsymbol{p}_1, \boldsymbol{p}_0, \boldsymbol{q}_1, \boldsymbol{q}_0) = P(\boldsymbol{q}_1, \boldsymbol{q}_0, \boldsymbol{p}_1, \boldsymbol{p}_0)$. This requirement deserves some deep explanation, to be corectly understood. The left side of the above equality is called the *cofactor* of P (written $cof(P)$) or also the *implicit* quantity index. By construction, it is such that $V = P \cdot cof(P)$, so it could be interpreted as a quantity index associated to P. On the other hand, another quantity index is naturally associated to P, by exchanging the role of the price and quantity vectors in the formula of P itself. This is the right side of the equality and is called the *correspondent* of P (written, $cor(P)$). In fact, if a function satisfies the mandatory axioms for price vectors, by exchanging price and quantity vectors we obtain a function satisfying the axioms for quantity indexes (which, as written above) are the same as those for price indexes, but where the roles of the price and quantity vectors are switched). So, in the end we have two quantity indexes naturally associated to a given price index and, by simmetry, we want them to be equal, i.e. we want $cof(P) = cor(P)$, which is a different way to state factor reversibility. Notice that the equality of the two induced quantity indexes is a property of index P and so this property has been introduced as a desirable requirement for price indexes. As in the basis reversibility case, also the factor reversibility axiom can be stated as an invariant property of the price index. Let us call $F(P) = V/cor(P)$ the *factor antithesis* of P. Then factor reversibility is the same as requiring $P = F(P)$.

As a matter of fact, and differently from the simple index case, from the above axioms it is not possible to derive a unique optimal price index and, in general, the axioms are used more as criteria to disregard unacceptable formuals. There is a number of price indexes proposed in literature, but in practice just a few are used, namely the Laspeyres index, the Paasche index and the Fisher index.

Remark. As it can be noticed, there is no associativity axiom, in the list. In principle, associativity is an important property for composite indexes too, but its technical definition in literature is currently not so neat. So to avoid difficulties we skip it, also because, as it will be clear in the following, the choice of the indexes to adopt is, in practice, based on other criteria not involving associativity, as a primary requirement.

Laspeyres price index. The Laspeyres price index compares the value of the basket at t_0, at the prices of t_0 and t_1, in formulas:

$$P_{lsp} = \frac{\sum_{1=1}^{n} p_{1i} q_{0i}}{\sum_{1=1}^{n} p_{0i} q_{0i}}. \tag{22}$$

In practice, it compares the expenditure you should make at t_1, to purchase the same basket of goods you purchased at t_0. The Laspeyres index is proportional, commensurable, homogeneous and monotonic (these properties can be checked directly). It is neither basis reversible, nor factor reversible. By direct calculation, one sees that its basis and factor antitheses coincide and are equal to the Paasche price index (see next item):

$$B(P_{lsp}) = F(P_{lsp}) = P_{psh}. \tag{23}$$

Checking the above equalities is straighforward. For example:

$$B(P_{lsp}) = \left(\frac{\sum_{1=1}^{n} p_{0i} q_{1i}}{\sum_{1=1}^{n} p_{1i} q_{1i}} \right)^{-1} = \frac{\sum_{1=1}^{n} p_{1i} q_{1i}}{\sum_{1=1}^{n} p_{0i} q_{1i}} = P_{psh}. \tag{24}$$

$$F(P_{lsp}) = \frac{\sum_{1=1}^{n} p_{1i} q_{1i}}{\sum_{1=1}^{n} p_{0i} q_{0i}} \Big/ \frac{\sum_{1=1}^{n} q_{1i} p_{0i}}{\sum_{1=1}^{n} q_{0i} p_{0i}} = \frac{\sum_{1=1}^{n} p_{1i} q_{1i}}{\sum_{1=1}^{n} p_{0i} q_{1i}} = P_{psh}. \tag{25}$$

Paasche price index. The structure of the Paasche index is analogous to that of the Laspeyres one, but with the quantities referring to t_1:

$$P_{psh} = \frac{\sum_{1=1}^{n} p_{1i} q_{1i}}{\sum_{1=1}^{n} p_{0i} q_{1i}}. \tag{26}$$

The Paasche price index compares the value of the basket at time t_1, to the value of the same basket, with t_0 prices. As the Laspeyres formula, it fulfills the axioms of proportionality, commensurability, homogeneity and montonicity, but neither basis, nor factor reversibility. Simmetrically to the Laspeyres case, the basis and factor antitheses of the Paasche index coincide and are equal to the Laspeyres price index:

$$B(P_{psh}) = F(P_{psh}) = P_{lsp}. \tag{27}$$

Fisher price index. The Fisher index is the geometric mean (in this context also known as the *crossing*) of the Laspeyres and the Paasche indexes:

$$P_{fsh} = \sqrt{P_{lsp} \cdot P_{psh}}. \tag{28}$$

The Fisher index is proportional, commensurable, homogeneous and monotonic, since the Laspeyres and Paasche indexes fulfill these same axioms. In addition, since the Laspeyres and the Paasche indexes are each the basis and the factor antitheses of the other, it can be easily proved that the index is both basis and factor reversible. For this reason, the Fisher index is called *ideal*.

Up to know, we have discussed price and quantity indexes separately. But in fact, when a price index formula P is selected, a quantity index is induced, namely the cofactor of P. So, price and quantity indexes go in pairs and one element of the pair is not axiomatically acceptable, unless also the other is. Unfortunately, in general a "good" price index need not induce a "good" cofactor. In particular, the cofactors of proprtional price indexes need not be proportional with respect to quantities, i.e. they cannot even be interpretable as quantity indexes. So in general, one must explicitly check whether the cofactor is, at least, proportional, this being the fundamental property specifying the nature of the index. As a matter of fact, the cofactors of Laspeyres, Paasche and Fisher indexes satisfy all of the mandatory requirements (and, in the case of the Fisher index, also the desirable ones), so those price indexes can be "safely" adopted.

3.3 Multitemporal price comparisons

In practice, we are not interested in comparing just two set of prices, but to follow price dynamics over longer time spans. This leads to the problem of multitemporal price comparison.

Suppose we measure the prices and the quantities of consumed goods at times $t_0, t_1, \ldots, t_h$. What we would like to achieve is a system of price comparisons P_{ij}, between pairs of times (i, j), whose elements are "axiomatically acceptable" and constitutes a *transitive* system, i.e. such that $P_{ik} = P_{ij}P_{jk}$ or, equivalently, $P_{ik} = P_{ij}/P_{kj}$. Unfortunately, this is not possible, since there is a conflict between transitivity and other axiomatic properties. More formally, let $P(\boldsymbol{p}_1, \boldsymbol{p}_0, \boldsymbol{q}_1, \boldsymbol{q}_0)$ be a price index satisfying the proportionality axiom and also the transitivity condition, so that

$$P(\boldsymbol{p}_2, \boldsymbol{p}_0, \boldsymbol{q}_2, \boldsymbol{q}_0) = P(\boldsymbol{p}_2, \boldsymbol{p}_1, \boldsymbol{q}_2, \boldsymbol{q}_1)/P(\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{q}_0, \boldsymbol{q}_1). \tag{29}$$

The left hand side of this expression does not depend upon $\boldsymbol{p}_1$ and $\boldsymbol{q}_1$, while the right side does. As a consequence

$$P(\boldsymbol{p}_2, \boldsymbol{p}_0, \boldsymbol{q}_2, \boldsymbol{q}_0) = \frac{P(\boldsymbol{p}_2, \boldsymbol{p}_1, \boldsymbol{q}_2, \boldsymbol{q}_1)}{P(\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{q}_0, \boldsymbol{q}_1)} = \frac{P(\boldsymbol{p}_2, \boldsymbol{p}_1^*, \boldsymbol{q}_2, \boldsymbol{q}_1^*)}{P(\boldsymbol{p}_0, \boldsymbol{p}_1^*, \boldsymbol{q}_0, \boldsymbol{q}_1^*)}, \tag{30}$$

for any fixed $\boldsymbol{p}_1^*$ and $\boldsymbol{q}_1^*$. In other words, $P(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{q}_1, \boldsymbol{q}_2)$ and $P(\boldsymbol{p}_0, \boldsymbol{p}_2, \boldsymbol{q}_0, \boldsymbol{q}_2)$ are only functions of $\boldsymbol{p}_2$, $\boldsymbol{q}_2$ and $\boldsymbol{p}_0$, $\boldsymbol{q}_0$ respectively, so that:

$$P(\boldsymbol{p}_1, \boldsymbol{p}_0, \boldsymbol{q}_1, \boldsymbol{q}_0) = \frac{F(\boldsymbol{p}_1, \boldsymbol{q}_1)}{F(\boldsymbol{p}_0, \boldsymbol{q}_0)}. \tag{31}$$

By the proportionality of P with respect to prices, we have, for any $\boldsymbol{p}_0$

$$1 = P(\boldsymbol{p}_0, \boldsymbol{p}_0, \boldsymbol{q}_1, \boldsymbol{q}_0) = \frac{F(\boldsymbol{p}_0, \boldsymbol{q}_1)}{F(\boldsymbol{p}_0, \boldsymbol{q}_0)} \tag{32}$$

or

$$F(\boldsymbol{p}_0, \boldsymbol{q}_0) = F(\boldsymbol{p}_0, \boldsymbol{q}_1), \tag{33}$$

so that function F cannot depend upon quantities (in fact, the above equality is an identity, i.e. it holds for any vector $\boldsymbol{p}_0$). In the end, we conclude that a transitive and proportional price index $P(\boldsymbol{p}_1, \boldsymbol{p}_0, \boldsymbol{q}_1, \boldsymbol{q}_0)$ is in fact just a function of the prices:

$$P(\boldsymbol{p}_1, \boldsymbol{p}_0, \boldsymbol{q}_1, \boldsymbol{q}_0) = G(\boldsymbol{p}_1, \boldsymbol{p}_0). \tag{34}$$

So, in general, a transitive and proportional price index must be a function of the compared price vectors and, possibly, of other parameters $\boldsymbol{q}_*$, but not of the quantities of the compared times. The final consequence is that *the cofactor of the price index cannot be proportional with respect to quantities.* To see this, consider that proportionality (with respect to quantities) of the cofactor of the price index implies

$$1 = \frac{\sum_1^n p_{1i} q_0}{\sum_1^n p_{0i} q_0} / P(\boldsymbol{p}_1, \boldsymbol{p}_0, \boldsymbol{q}_*) \Rightarrow \frac{\sum_1^n p_{1i} q_0}{\sum_1^n p_{0i} q_0} = P(\boldsymbol{p}_1, \boldsymbol{p}_0, \boldsymbol{q}_*) \tag{35}$$

which is impossible, since one side depends upon $\boldsymbol{q}_0$, while the other does not.

So, we cannot build systems of price comparisons composed of transitive indexes with proportional cofactors (i.e. price index whose implied quantity indexes satisfy the fundamental property of proportionality with respect to quantities). The way out to this problem is to build the final price comparison systems, by means of two different kinds of price indexes:

1. A set of *direct* bilateral indexes, whose cofactors satisfy the proportionality axiom.
2. A set of *indirect* indexes, derived from the direct ones, which are transitive, but whose cofactors are not proportional.

The idea is that we should preserve cofactor proportionality when comparing situations with similar quantity structures and that we may relax it, when quantity structures are very different. In practice, this leads to using direct price indexes for situations near in time, and indirect indexes for situations temporally more distant (since we assume that quantity structures are more similar for nearer times).

To clarify how this idea works, suppose we want to compare prices over four times $t_1, \ldots, t_4$ and suppose we have both price and quantity vectors at each time. We can build the direct Fisher indexes F_{21}, F_{32}, F_{43} and analogously F_{12}, F_{23}, F_{34} (which are also equal to the reciprocals of the previous ones). Then. instead of computing direct indexes between the other time pairs, from F_{21}, F_{32}, F_{43} we compute the indirect transitive indexes $F_{31} = F_{32}F_{21} = F_{32}/F_{12}$, $F_{41} = F_{42}F_{21} = F_{42}/F_{12}$ and so on…, getting a subset of transitive price comparisons.

In practice, the kind of indexes and the way these are combined into a transitive system depends upon the data which are actually available. There are three main possibilities:

1. **Fixed basis**. We have the price vectors for all the compared times, but the quantity vector for (say) only t_0, which is then assumed as the *basis* of the comparison. In this case, all we can do is to build direct Laspeyres price indexes L_{x0}, their reciprocals $P_{0x} = 1/L_{x0}$ and transitive price indexes of the form $P_{xy} = L_{x0}/L_{y0}$.

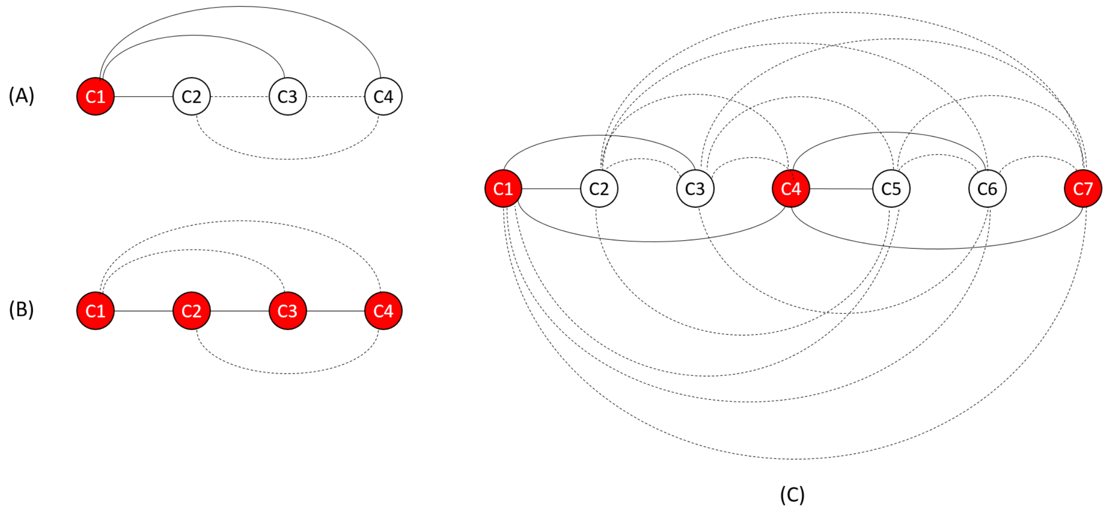


Figure 1: (A) Fixed basis system; (B) Moving basis system; (C) Periodic basis system.

2. **Moving basis**. This is the case described above, where we know price and vectors at each time, so that time t_i can be used as basis for its neighbor times.

3. **Periodic basis**. Since surveying quantities (e.g. consumed) is an expensive and difficult task, the typical situation is to have price data for each time t_i and quantity data for times $t_0, t_{0+k}, t_{0+2k} \ldots$ In other words, we have prices for each time and quantities on a longer period. As a consequence, one assumes the times when quantities are surveyed as bases and builds:

   - Direct price comparisons between a time t_i and its more recent basis b_i, using Laspeyres indexes, $P_{ib_i} = L_{ib_i}$.
   - Their reciprocals $P_{b_i i} = 1/L_{ib_i}$.
   - Chain indexes between subsequent pair of bases $C_{b_{i+k}b_i} = L_{b_{i+k}b_i}$, $C_{b_i b_{i+k}} = 1/C_{b_{i+k}b_i}$ $(i = 0, k, 2k \ldots)$.
   - Indirect price indexes $P_{ij} = L_{ib_i} C_{b_i b_j} L_{b_j j}$, between the other pairs of times.

The above cases are graphically depicted in Figure 1.

4 Examples of price indexes and price comparisons

4.1 The measurement of consumer inflation in Italy

To measure and monitor the dynamics of consumption prices, the Italian National Statistical Bureau produces three different consumption price indexes, namely the **NIC** index (national consumner price index, for the whole population), the **FOI** index (index for workers' and employees' families) and the **IPCA** index (harmonised national consumer price index, for comparisons at European level). The three indexes have similar structures, but have different aims. The NIC index measures inflation at the level of the overall economic systems, as the entire consumer population would be just a single entity (so, overlooking all of the differences in consumption behaviors of different subjects), and is the main price index in view of economic policies. The FOI index refers to a specific subset of people and is instead used for tuning rents and some kinds of monetary subsidia. It is computed in two "versions", i.e. *with tobacco* and without *tobacco*. The IPCA index is built similarly to the NIC index and is used to monitor the convergence of European economies.

The computation structures of the indexes are quite similar (and follow international standards), but there are nevertheless some differences. The NIC and the FOI share the same basket of goods/services, but adopts different weights, to reflect the different reference populations. The IPCA index shares with the NIC the reference population, but excludes some goods from the NIC basket (e.g. lotteries) and considers prices actually paid by consumers (i.e. discounted prices, promotions or tickets for drugs).

The basket of goods and services is updated on a yearly basis, so as to follow the change in consumption behavior (notice, however, that this makes the comparison over time a subtle issue). To give an idea of the complexity of the computation procedure, in the 2017 basket for the NIC and the FOI indexes, there are 1481 elementary products, grouped in 920 products which, in turn, are clustered into 405 aggregates, after the EICOCOP (European Classification of Individual Consumption by Purpose) structure. Similarly, the IPCA index is based on a basket with 1498 elementary products, grouped into 923 products, clustered into 409 aggregates. All in all, each month about 706500 prices are collected in about 41700 selling points and 8000 houses (to get rent prices), located in 96 municipalities across Italy.

The practical computation of the NIC, FOI and IPCA is extremely complex, and will not be discussed in detail here. Basically, the computation consists of a hierarchy of aggregation, where the price variation of elementary products are combined together by the Jevons index and then progressively aggregated at provincial, regional and national level, using the Laspeyres formula, with weights (quantities) estimated by using data mainly from the System of National Accounts.

For each index, Istat provides monthly time series over the years, for different products, aggregates of products and for the overall indexes (see http://dati.istat.it/). The following picture depicts the monthly trajectories of the three indexes from January 2016 to April 2018. The basis is assumed to be year 2015, this means that the average that the basket value with the prices of month m is compared to the basket value with the average prices of year 2015, put conventionally to 100. As it can be seen, the NIC and the FOI have very similar trajectories, while the IPCA is quite different, as a consequence of the differences in the way it is built. Periodically, Istat changes the reference year and produces chain coefficients to link the indexes expressed with respect to different bases, as explained in the main text.

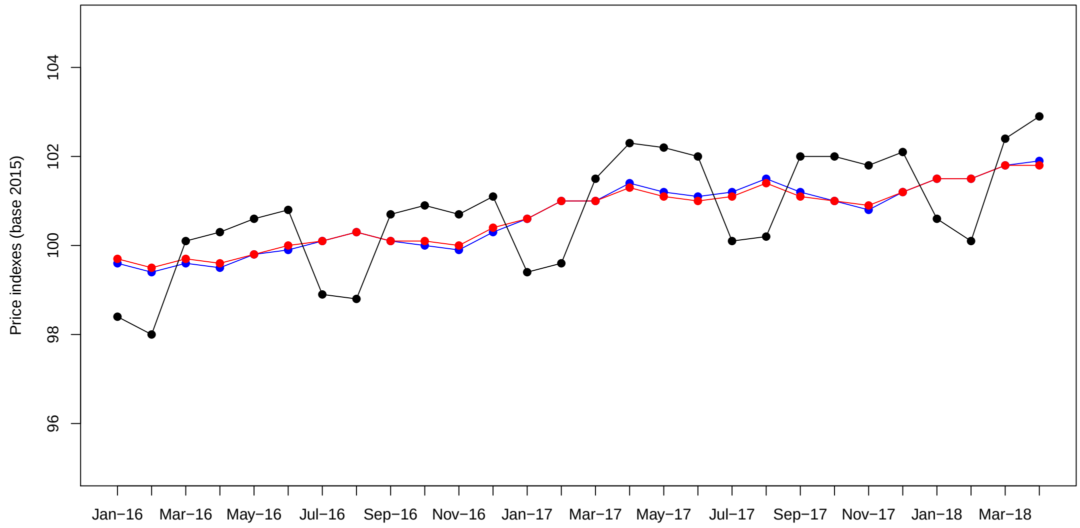


Figure 2: Monthly time series of NIC (blue), FOI (red) and IPCA (black) indexes from January 2016 to April 2018 (basis = year 2015).

4.2 Production indexes

In addition to prices at consumption level, National Statistical Offices measure also prices at *production* level, i.e. before other price components sum up, to form the final price. National production is articulated in a complex classification system (NACE 2007, take a look at http://dati.istat.it, look for "prices" and click on "NACE 2007", to realize the complexity of the classification). For each product, its production price (actually, there are more kinds of production prices. . .) is provided as a time series, deflated with respect to a fixed basis, and chain coefficients are also given, to link time series referring to different bases (in fact, it is actually a periodic basis system). To get a flavour of the data, look again at http://dati.istat.it.

4.3 Deflating the GDP

Given its importance for economic policies, monitoring real GDP is an essential actividty for economists (see Figure 3). In principle, this simply requires building a suitable price index P_{10}, to deflate the GDP at t_1 to the prices of t_0. At the same time, in the System of National Accounts, GDP is related to the total production and the intermediate goods ($GDP = PROD - INT$) and one could act indirectly, deflating the two entities on the right using their own price indexes P_{10}^{prod} and P_{10}^{int}. Writing real GDP as $\overline{GDP}$, we have:

$$\overline{GDP} = PROD/P_{10}^{prod} - INT/P_{10}^{int}. \tag{36}$$

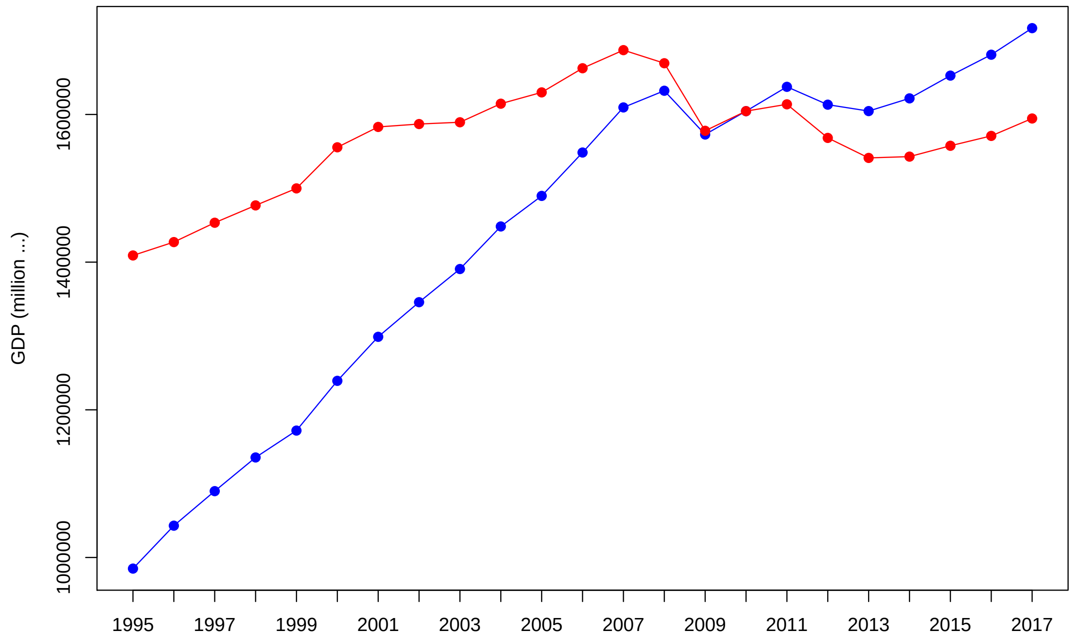


Figure 3: Annual Italian GDP 1995-2017 (current prices - blue; constant prices - red, basis = year 2010).

This procedure (which is that actually followed in practice) is called *double deflation*. As a result, the GDP is deflated with a price index which is an implicit mean of P_{10}^{prod} and P_{10}^{int}, i.e. by means of so-called *implicit deflator*, defined as $GDP/\overline{GDP}$.

5 Purchasing Power Parity and spatial price indexes (hints)

So far, we have discussed how to measure price variations over time; however we may also want to compare prices across spaces, for example to assess the differences in GDP per capita, in two or more different countries and, in general, to compare the internal *purchasing power* of the currencies. This leads to the topic of *spatial price indexes*, whose aim is to build so called *Purchasing Power Parities* (PPPs) among currencies (i.e. *real exchange rates*, so as to say) and which has some additional complications with respect to the temporal case. In fact, time is a completely ordered variable and, if time steps are not too long, the consumption or production structure does not change much from one point in time to the next, so that we can build a relatively smooth sequence of comparisons between "similar" price/quantity structures. On the contrary, in the spatial case we may have to compare completely different economic systems and there need not

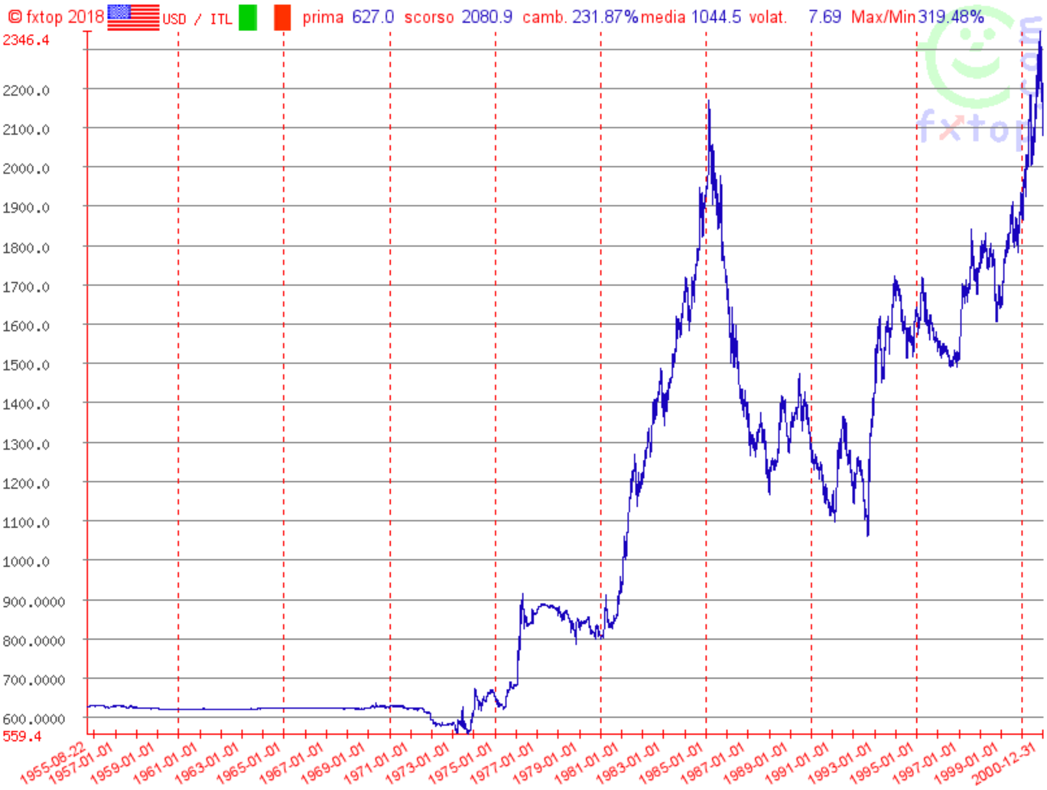


Figure 4: Exchange rates US Dollar - Italian Lira, 1953 - 2000 (http://fxtop.com/it).

be any natural "direction" driving us in building smoother comparisons (to understand the issue, try to compare your food expenditure, in terms of goods, with that of people from another continent. . .). So the first problem is that two countries need not share a common basket of goods or, more realistically, that they just share some "sub-basket"; moreover, in the spatial case we are mainly interested in multilateral comparisons, with the additional technical difficulties mentioned above, for multitemporal comparisons.

Exchange rates. One may wonder why not to use official exchange rates, to compare economic aggregates expressed in different currencies. The reason is that the way the rate is formed in the market does not reflect the internal purchasing powers of the compared currencies. The exchange rates mainly depends upon goods that are sold and purchased in international commercial activities. In addition, the exchange rates depend upon the actions of central banks and their fluctuations need not reflect changes in the purchasing powers (see Figure 4; can you guess what happened in 1973?). So even if exchange rates are very easy to use and produce transitive comparison systems, they are not suitable to compute PPPs and so to build tools to deflate spatially.

Using axiomatic price index theory. We can address the problem of computing

PPPs by using the results of price index theory that we have discussed before. Suppose we have to compare the prices of m countries. Let P_{ij} the price index comparing country j to country i (which is the basis of the comparison). What we would like to have is a system of transitive comparisons, i.e. such that $P_{ik} = P_{ij} \cdot P_{jk}$, and also such that the cofactor of each bilateral index is proportional. As we know, this is impossible and we have a trade-off between transitivity and cofactor proportionality. As in the multitemporal case, the way out to this problem is to make a compromise: we identify a subset of direct bilateral comparisons, using indexes with proportional cofactors, and a subset of indirect comparisons which are transitive, but whose cofactors are not proportional.

To clarify how to address this problem in the spatial case, let us consider the following example (see Table 1), where we have 4 countries and 8 goods. The columns of the table report the prices of the goods in each country. We see that only country 2 has a good basket *comparable* with those of the other countries (i.e. with enough common goods); in addition, we suppose to know only the quantities for country 2, provided in column Q2. Given these conditions, what we can do is to compare directly any country with country 2 (assumed as the basis) by a Laspeyres index and then to perform the other comparisons indirectly, as shown in Table 2 (the numerical results are reported in Table 3). By definition, the comparison of each country with itself is equal to 1; the direct bilateral indexes L_{12}, L_{32} and L_{42} and their reciprocals $P_{21} = 1/L_{12}$, $P_{23} = 1/L_{32}$ and $P_{24} = 1/L_{42}$ satisfy the mandatory axiomatic properties (in particular, their cofactors are proportional and can be actually interpreted as quantity indexes); the indirect indexes $P_{13} = L_{12}/L_{32}$, $P_{14} = L_{12}/L_{42}$ and the others are transitive by construction, but their cofactors are not proportional.

Table 1: Example of price and quantity data for multispatial comparisons.

	C1	C2	C3	C4	Q2
G1	2	-	-	-	-
G2	3	5	-	-	15
G3	2	6	-	4	20
G4	3	5	-	5	60
G5	-	6	10	6	20
G6	-	5	8	7	60
G7	-	7	12	-	10
G8	-	-	15	-	-

As it should be clear from the above example, the idea is to identify a minimal set of direct bilateral comparisons, on which to fulfill the mandatory requirements, and a set of indirect comparisons, derived from the direct ones, satisfying transitivity, but losing the proportionality of the cofactor. The practical way to achieve this depends upon the data at hand. In the example, we had just one possible country to select as the basis, since other good baskets have too small intersections and since we have just the quantities in

Table 2: Example of price index systems for multispatial comparisons.

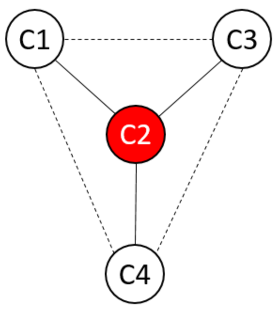
	C1	C2	C3	C4
C1	1	$1/L_{12}$	L_{32}/L_{12}	L_{42}/L_{12}
C2	L_{12}	1	L_{32}	L_{42}
C3	L_{12}/L_{32}	$1/L_{32}$	1	L_{42}/L_{32}
C4	L_{12}/L_{42}	$1/L_{42}$	L_{32}/L_{42}	1

Table 3: Values of the price indexes of Table 2 on the data from Table 1.

	C1	C2	C3	C4
C1	1	1.8679	3.0497	2.0458
C2	0.5354	1	1.6327	1.0952
C3	0.3279	0.6125	1	0.6708
C4	0.4888	0.9130	1.4907	1

country 2. Indeed, other situations can be possible. For example, we might have the quantities of more, if not all, countries and/or it might be that the structure of basket comparabilities is more complex. Let us turn this discussion into graph terms (see Figure 5). Similarly to the multitemporal case, we represent each country as a *node* and we add an edge (as a solid line) when the respective baskets are comparable. To each solid edge corresponds also a direct bilateral comparison, while indirect transitive comparisons are depicted as dashed lines. The graph on the left represent the above described example, where a single country acts as a common basis and where we can only use Laspeyres indexes. The graph on the right represents a situation where different cases of bilateral comparisons can be built, then generating a system of indirect transitive comparisons. Based on the available information on quantities, bilateral comparisons can be based on Laspeyres, Paasche or Fisher formulas, while the indirect ones are computed, as usual, as ratios (or products) of the direct indexes.

The system of comparisons outlined above is called *open*, since adding a new country z to



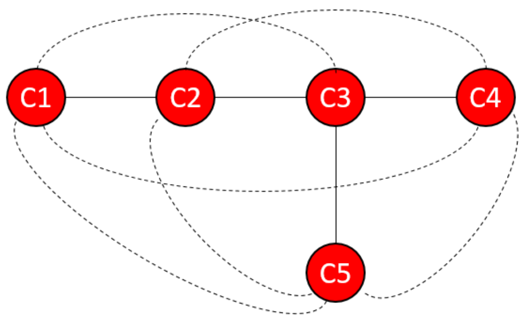


Figure 5: Examples of multispatial price comparison graphs.

the set of compared countries does not affect the previous comparisons (in other words, the price index P_{ij} is not affected by the introduction of country z). An alternative is to build *closed* systems. i.e. to develop price index systems which are forced to be transitive at the cost of (i) losing cofactor proportionality and (ii) building a system of comparisons which must be recalculated each time a new country is added or removed. In fact, the easiest way to achieve transitivy, in the comparison of m countries $1, \dots, n$ (on a basket of k goods) is to build bilateral price indexes which depend upon the prices of the two compared countries and on virtual quantities, common to all of the m countries. In practice, the bilateral index comparing country i and country j (the basis) has typically the following structure:

$$P(\boldsymbol{p}_i, \boldsymbol{p}_j, \hat{\boldsymbol{q}}) = \frac{\sum_{s=1}^{k} p_{is}\hat{q}_s}{\sum_{s=1}^{k} p_{js}\hat{q}_s}, \tag{37}$$

where the quantity vector $\hat{\boldsymbol{q}}$ is common to the compared countries. Such indexes are trivially transitive, in fact:

$$P(\boldsymbol{p}_i, \boldsymbol{p}_j, \hat{\boldsymbol{q}}) = \frac{\sum_{s=1}^{k} p_{is}\hat{q}_s}{\sum_{s=1}^{k} p_{js}\hat{q}_s} = \frac{\sum_{s=1}^{k} p_{is}\hat{q}_s}{\sum_{s=1}^{k} p_{ts}\hat{q}_s} \cdot \frac{\sum_{s=1}^{k} p_{ts}\hat{q}_s}{\sum_{s=1}^{k} p_{js}\hat{q}_s} = P(\boldsymbol{p}_i, \boldsymbol{p}_t, \hat{\boldsymbol{q}}) \cdot P(\boldsymbol{p}_t, \boldsymbol{p}_j, \hat{\boldsymbol{q}}).$$

The quantity vector $\hat{\boldsymbol{q}}$ can be built in different ways, producing differenr close systems. In literature, one may find the following three main proposals:

1. **ECLA** (Economic Commision of Latin America): $\hat{q}_i = \sum_{i=1}^{m} q_i$, i.e. the common quantity of a good is the sum of the quantities of that good, in each country (or, equivalently, the arithemtic mean).

2. **Gerardi UCW** (Unit-Country-Weight): $\hat{q}_i = \left(\prod_{i=1}^{m} q_i\right)^{1/m}$, i.e. the common quantity of a good is the geometric mean of the quantities of that good, in each country.

3. **Geary-Khamis**: the Geary-Khamis price index P_{ij}^{GK} between two countries i and j is defined as the cofactor of the quantitiy Geary-Khamis index Q_{ij}^{GK}:

$$P_{ij}^{GK} = \frac{V_{ij}}{Q_{ij}^{GK}}.$$

The Geary-Khamis quantity index, in turn, is computed as a function of both the prices and the quantities of all of the countries in the system. Formally:

$$Q_{ij}^{GK} = \frac{\sum_{s=1}^{k} \pi_s q_{is}}{\sum_{s=1}^{k} \pi_s q_{js}}$$

where the virtual prices are defined by:

$$\pi_s = \sum_{\ell=1}^{m} c_\ell p_{\ell s} \frac{q_{\ell s}}{\sum_{\ell=1}^{m} q_{\ell s}}$$

and

$$c_\ell = \sum_{s=1}^{k} \pi_s \frac{q_{\ell s}}{\sum_{s=1}^{k} p_{\ell s} q_{\ell s}}$$

Remark. The virtual price of good s is computed as a weighted average of the price of that good in all of the compared countries. Each price is converted to an "international unit of measure", by a coefficient c_ℓ, which depends only upon the country, which is calculated by the last equation, again as a function of the virtual prices. So the Geary-Khamis index is obtained by a system of equation that must be solved jointly.

From the above examples, it is clear that when a new country is added to the comparison, the virtual quantities must be re-computed, so affecting the previously built price indexes.

A different way to build closed systems, without using virtual quantities, is provided by the so-called EKS index, which is basically built as the transitive index system which best approximates a given system of direct bilateral indexes. Suppose we have both the prices and the quantities of the same basket, in m countries, so that we can build a system of $m(m-1)$ bilateral Fisher indexes (which do not fulfill transitivity). Each country s can be assumed as the basis for a set of $m-1$ Fisher price indexes F_{is} $(i \neq s)$, so that we can also build indirect price indexes $P_{ij}^{(s)} = F_{is}/F_{js}$. Since no country has any privileged role, in the system (recall that we have prices and quantities for all countries), we "symmetrize" indexes $P_{ij}^{(s)}$, by computing

$$P_{ij} = \left(\prod_{s=1}^{m} P_{ij}^{(s)} \right)^{1/m}, \tag{38}$$

getting a system of geometric means of bilateral indexes (the EKS system) which do satisfy transitivity.

All in all, we see how the multispatial comparison problem can be addressed in a number of different ways, each of which has its advantages and its drawbacks. Clearly, using different approaches, one gets also to different results (see Table 4), so making it evident how spatial price comparison is a highly subtle and problematic issue.

Table 4: Consumption spatial price indexes for some European countries (year 1990; basis=Germany).

Method	D	B	NL	F	UK	I
ECLA	100	97.1	94.9	96.7	84.6	92.1
Geary-Khamis	100	93.3	89.6	95.8	81.1	89.8
Gerardi UCW	100	97.7	95.4	96.4	85.1	90.7
EKS	100	94.9	92.4	96.3	83.2	90.2